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The Dilation Factor of the Peano–Hilbert Curve

K. E. Bauman

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Abstract—It is proved that the maximum value of the ratio $|p(x) - p(y)|^2/|x - y|$ for the Peano–Hilbert curve $p\colon [0, 1] = I \to I^2$ is equal to 6.

Key words: *square-to-linear ratio*, *square-to-linear dilation factor*, *Peano–Hilbert curve*.

INTRODUCTION

In this paper, we determine the maximum value of the *square-to-linear ratio*

$$\frac{|p(x) - p(y)|^2}{|x - y|}$$

for the classical Peano–Hilbert curve $p\colon I \to I^2$, which maps the unit interval $I = [0, 1]$ to the unit square.

In the first part of the paper, we specify pairs of points for which the square-to-linear ratio is equal to 6. In the second part, we prove that the square-to-linear ratio is at most 6 for any pair of points of the Peano–Hilbert curve.

Peano curves are applied in encoding of information (see [1]), numerical integration, and other applications of mathematics. The *square-to-linear dilation factor*, which is defined as the maximum square-to-linear ratio over all pairs of points, is an important characteristic of Peano curves. It was shown in [2, 3] that of most interest in applications are curves with minimum square-to-linear dilation factor.

For an explicitly specified fractal Peano curve, it is easy to calculate the square-to-linear dilation factor by computer; however, the theoretical proof that the computed value is indeed the largest square-to-linear ratio involves significant difficulties.

This paper contains the first such proof. The result is related to the Peano–Hilbert curve, which is the simplest and best-known Peano curve. In [2], Peano curves with square-to-linear ratios less than 6 were constructed. Presumably, the minimal curves of fractal genus 9 constructed there have square-to-linear ratio $5\frac{2}{3}$. But this result has not been substantiated theoretically. The technique developed by the author in this paper uses the high-degree symmetry of the Peano–Hilbert curve and cannot be carried over to the less symmetric curves considered in [2].

The main result of this paper is the following theorem.

Theorem 1. *The square-to-linear dilation factor of the Peano–Hilbert curve is equal to* 6.

The Peano–Hilbert curve. We start the construction of a map $p\colon I \to Q$ from a given interval I to a given square Q by defining the value p at the center (midpoint) of the interval I to be the center (of symmetry) of the square Q.

Let us divide the interval I into four equal intervals I_1, I_2, I_3, and I_4 and the square Q into four equal squares Q_1, Q_2, Q_3, and Q_4. These four intervals are called the *fractions* of the initial interval; they are numbered so that fractions with larger numbers are nearer to the endpoints of the interval. In particular, the leftmost point of the interval belongs to I_1, and $I_k \cap I_{k+1} \neq \varnothing$ for

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$k = 1, 2, 3$. The constructed squares are also called the *fractions* of the initial square and numbered so that Q_k has a common edge with Q_{k+1} for $k = 1, 2, 3$.

The map p is defined at the centers of the fractions of the interval so that the image of the center of I_k coincides with the center of Q_k for $k = 1, 2, 3, 4$.

Then, we divide each fraction I_k of the initial interval into four equal fractions $I_{k,1}$, $I_{k,2}$, $I_{k,3}$, and $I_{k,4}$, which are called the *second-order fractions* and are numbered in the same way as the first-order fractions. Note that $I_{k,4} \cap I_{k+1,1} \neq \varnothing$ for $k = 1, 2, 3$.

Similarly, Q_k is partitioned into four second-order fractions $Q_{k,1}$, $Q_{k,2}$, $Q_{k,3}$, and $Q_{k,4}$ numbered by the same rule as above and satisfying the additional condition $Q_{k,4} \cap Q_{k+1,1} \neq \varnothing$ for $k = 1, 2, 3$, which can always be ensured thanks to the freedom of choice of the first element. After the second subdivision, the map p is defined at the centers of all second-order fractions; namely, it takes the center of each $I_{j,k}$ to that of $Q_{j,k}$, i.e., of the second-order fraction of the square with the same number.

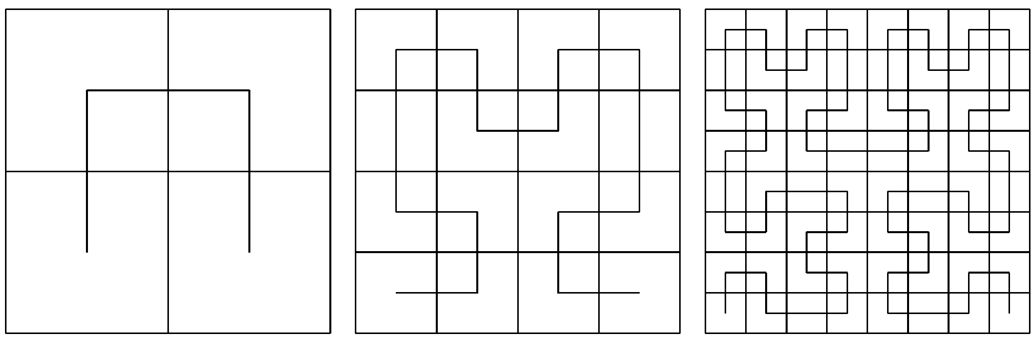

Fig. 1. Three steps of the construction of the Peano–Hilbert curve

Then, we subdivide the second-order fractions into third-order fractions, the third-order fractions into fourth-order fractions, and so on. After infinitely many steps, we define the map p on the everywhere dense set of the centers of fractions of all orders, which has a unique extension to a continuous map from the initial interval to the square. The map thus constructed takes each fraction of the interval to the corresponding fraction of the square (of the same order).

Linearly interpolating the map of the centers of kth-order fractions defined above, we obtain curves which approximate the Peano–Hilbert curve up to the radius of the circle circumscribed about a kth-order fraction. The interpolation curves for the first three steps of the construction are shown in Fig. 1.

According to [2], the functional equation of the unit Peano–Hilbert curve (which maps the unit interval to the unit square) in the complex plane can be is written as

$$2p(t) = \begin{cases} i\overline{p(4t)} & \text{for } \ 0 \le t \le 1/4, \\ i + p(4t-1) & \text{for } \ 1/4 \le t \le 1/2, \\ i + 1 + p(4t-2) & \text{for } \ 1/2 \le t \le 3/4, \\ i + 2 - i\overline{p(4t-3)} & \text{for } \ 3/4 \le t \le 1 \end{cases} \tag{1}$$

(the bar denotes complex conjugation and i is the imaginary unit). The Peano–Hilbert curve is the unique continuous solution of Eq. (1) and the unique regular Peano curve of fractal genus 4 in the sense of [3].

1. A LOWER BOUND

We regard the interval on which the map p is defined as the unit time interval. Then the curve is the trajectory of a point during unit time. Accordingly, the preimages of the points of the curve can be regarded as the moments of time at which these points are passed.

Lemma 1. *The unit Peano–Hilbert curve with starting point at the lower left corner of the square passes its upper left corner at the moment* $1/3$, *the upper right corner at the moment* $2/3$, *and the lower right corner at the moment* 1 (see Fig. 2).

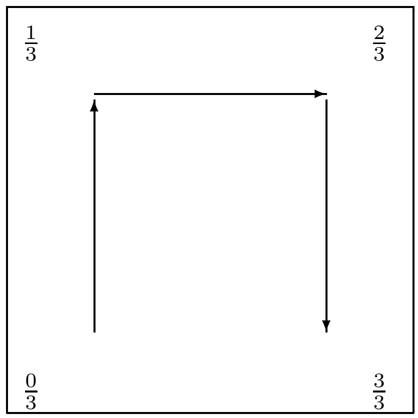

Fig. 2. The moments of time at which the corners are passed

Proof. The functional equation (1) implies

$$\begin{aligned} 2p(0) &= i\overline{p(0)} &&\Longrightarrow\ p(0) = 0, \\ 2p\Big(\frac{1}{3}\Big) = i + p\Big(\frac{4}{3} - 1\Big) &= i + p\Big(\frac{1}{3}\Big) &&\Longrightarrow\ p\Big(\frac{1}{3}\Big) = i, \\ 2p\Big(\frac{2}{3}\Big) = i + 1 + p\Big(\frac{8}{3} - 1\Big) &= i + 1 + p\Big(\frac{2}{3}\Big) &&\Longrightarrow\ p\Big(\frac{1}{3}\Big) = i + 1, \\ 2p(1) = i + 2 - i\overline{p(4-3)} &= i + 2 - i\overline{p(1)} &&\Longrightarrow\ p(1) = 1. \quad \square \end{aligned}$$

Lemma 2. *The square-to-linear dilation factor of the unit Peano–Hilbert curve is at least* 6.

Proof. The functional equation of the Peano–Hilbert curve implies the self-similarity of this curve; that is, the restriction of the curve to any fraction is similar to the entire curve. Therefore, by virtue of Lemma 1, the moments of passing the corners of the fractions of the unit square are obtained by dividing the time intervals during which these fractions are traversed into three parts.

The moments of passing the corners of the fractions of the unit square thus determined are shown in Fig. 3. The numbers in this figure are the moments of passing the corners of the corresponding fractions (some corners, which are shared by different fractions, are passed several times). Calculating the square-to-linear ratio for all pairs of corner points, we find the maximal ratio; this is a lower bound for the square-to-linear dilation factor.

At every step of the construction of the curve, the number of points passed at known moments of time is quadrupled. Calculating them, we obtain points passed at the moments $t_1 = 23/48$ and $t_2 = 25/48$ (see Fig. 4) for which the square-to-linear relation is equal to 6 already at the fourth step:

$$\frac{(1/2)^2}{25/48 - 23/48} = 6. \quad \square$$

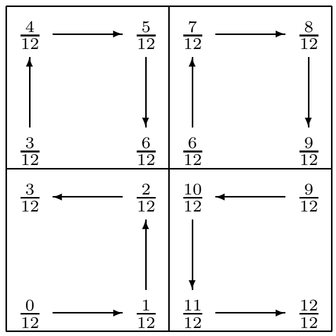

Fig. 3. The moments of passing the corners of fractions

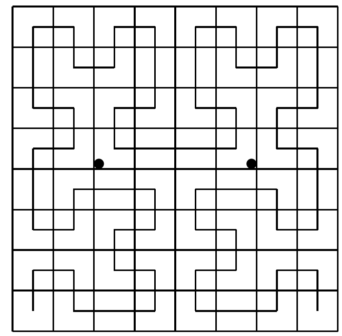

Fig. 4. A pair of points for which the square-to-linear ratio is equal to 6

2. AN UPPER BOUND

The division of an interval into three equal parts determines a division of the square which is the image of this interval under the Peano–Hilbert map into three parts, which we call the *thirds* of the square. The first third of the square is the image of the first third of the interval.

Lemma 3. *Suppose that the corners of the unit square are passed in the following order*: *upper left*, *lower left*, *lower right*, *upper right. Then the first third of the square is the union of an infinite sequence of square fractions* Q_k $(k = 1, 2, \dots)$ *such that*

$$Q_k = \left\{ (x, y) \mid 0 \le x \le \frac{1}{2^k},\ \left(1 - \frac{1}{2^{k-1}}\right) \le y \le \left(1 - \frac{1}{2^k}\right) \right\},$$

where x denotes the distance to the left edge of the square and y denotes the distance to the its upper edge (see Fig. 5). *Every such square Q_k is the image of the time interval*

$$\left[\frac{1}{3}\left(1 - \frac{1}{4^{k-1}}\right), \frac{1}{3}\left(1 - \frac{1}{4^k}\right) \right].$$

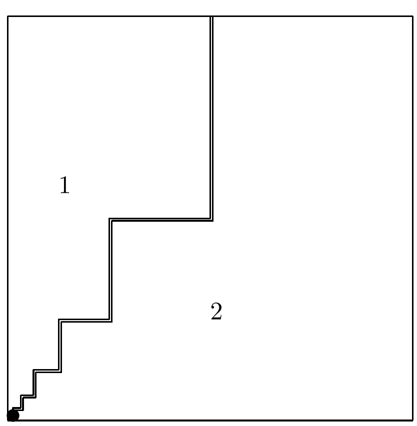

Fig. 5. A third and its complement

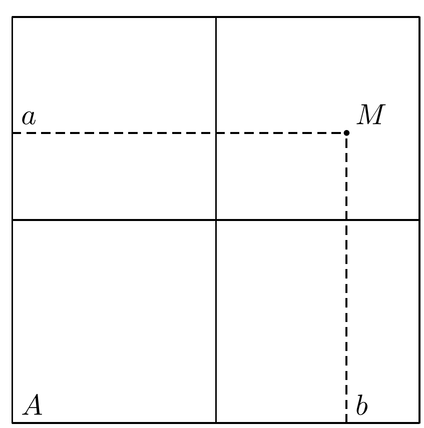

Fig. 6. Estimation of time

Proof. The square Q_1 is the first quarter of the unit square, and the first third of the unit square is, obviously, the union of Q_1 and the first third of the second quarter of the unit square. Next, the first third of the second quarter consists of Q_2, which is the first quarter of the second quarter of the unit square, and the first third of the second quarter of the second quarter. Similarly, the first third contains Q_3 and the remaining fractions Q_k. Since the Q_k are traversed in the order of increasing indices and the time of traversing each Q_k is $1/4^k$, it follows that the moment of time at which Q_k is passed is determined by

$$\frac{1}{3}\left(1-\frac{1}{4^k}\right)=\frac{1}{4}+\frac{1}{4^2}+\dots+\frac{1}{4^k}, \tag{2}$$

which proves the assertion concerning the time interval during which Q_k is traversed. For $k=\infty$, this inequality implies that the time required to traverse all of the fractions Q_k is $1/3$. □

Lemma 4. *Let $M(b,a)$ be a point for which $1/2^{n+1}<a\le 1/2^n$. Then the time T required for the Peano–Hilbert curve to go from $A(0,0)$ to M satisfies the inequality*

$$T\ge\frac{1}{3}\,\frac{a}{2^n}.$$

Proof. (1) Suppose that $a=1/2^n$. Without lost of generality, we can assume that $a\ge b$ (otherwise, we interchange a and b). Let us measure the time required to go from A to the upper edge of the nth-order square fraction, which has side $1/2^n$ (see Fig. 6). The point on this edge that is passed first is the upper left corner. According to Lemma 1, the time needed to go from A to this corner is one third of the time needed to traverse the entire square, which equals $1/4^n$. The point M cannot be reached in less time, i.e.,

$$T\ge\frac{1}{3}\,\frac{1}{4^n}\ge\frac{1}{3}\,\frac{1}{2^n}a.$$

The same estimate is valid when M does not belong to the first third of the curve fraction under consideration.

(2) Suppose that $1/2^{n+1}<a<1/2^n$ and M belongs to the first third of the curve fraction under consideration. Then there exists a positive integer $m>1$ for which

$$\frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}+\dots+\frac{1}{2^{n+m-1}}\le a<\frac{1}{2^{n+1}}+\dots+\frac{1}{2^{n+m-1}}+\frac{1}{2^{n+m}}. \tag{3}$$

By virtue of Lemma 3, this means that reaching the point M from A requires traversing each of the squares with edges $1/2^{n+1},1/2^{n+2},\dots,1/2^{n+m-1}$ at least once. Therefore, the time needed to reach M is at least the total time needed to traverse these squares, i.e.,

$$T\ge\frac{1}{4^{n+1}}+\frac{1}{4^{n+2}}+\dots+\frac{1}{4^{n+m-1}}=\frac{1}{4^{n+1}}\,\frac{1-1/4^m}{1-1/4}=\frac{1}{3}\frac{1}{4^n}\left(1-\frac{1}{4^m}\right).$$

It follows from (3) that

$$a<\frac{1}{2^{n+1}}\,\frac{1-1/2^{m+1}}{1-1/2}=\frac{1}{2^n}\left(1-\frac{1}{2^{m+1}}\right)\le\frac{1}{2^n}\left(1-\frac{1}{4^m}\right).$$

The last two inequalities imply

$$\frac{1}{3}\,\frac{1}{2^n}a<T. \quad \square$$

Corollary 1. *If the distance from some point to an edge of the unit square is equal to $a\ge 1/2$, then the time T necessary for the Peano–Hilbert curve to reach this edge from the given point satisfies the inequality $T\ge a/3$.*

We denote the square-to-linear ratio for a pair of moments of time x,y by

$$F(x,y)=\frac{|p(x)-p(y)|^2}{|x-y|}.$$

Lemma 5. *If $p(x)$ and $p(y)$ belong to fractions K_x and K_y of the same order which intersect in one common corner, then $F(x,y) \le 6$.*

Proof. Since the "scaled" curve $kp(x/k^2)$ has the same square-to-linear dilation as the unit Peano–Hilbert curve $p(x)$ for any positive k, we can take the scale $k = 1/2^n$, so that the squares K_x and K_y are unit and traversed in unit time.

We assume that the upper right corner of K_x coincides with the lower left corner of K_y. Let Q_x denote the quarter of K_x containing the lower left corner of K_x, and let Q_y denote the quarter of K_y containing the upper right corner of K_y (see Fig. 7).

There are three possible cases for the position of $p(x)$ and $p(y)$ with respect to these quarters.

The first case is $p(x) \notin Q_x$ and $p(y) \notin Q_y$. In this case, the most remote points of the squares K_x and K_y with the quarters Q_x and Q_y removed are the lower right corner of Q_x and the upper left corner of Q_y; the squared distance between them is equal to $1^2 + 2^2 = 5$. Since $|x - y| > 1$, it follows that $F(x,y) < 5$ in this case.

The second case is $p(x) \in Q_x$ and $p(y) \notin Q_y$. In this case, the distance between $p(x)$ and $p(y)$ does not exceed the distance between the lower left corner of K_x and the upper left corner of Q_y, which is equal to the square root of $1.5^2 + 2^2 = 6.25$. Since the time required to leave K_x starting from x is at least 0.25, it follows that $|x - y| \ge 1.25$ and $F(x,y) \le 6.25/1.25 = 5$. The case in which $p(x) \notin Q_x$ and $p(y) \in Q_y$ is similar.

The third case is $p(x) \in Q_x$ and $p(y) \in Q_y$. In this case, obviously, $|p(x) - p(y)|^2 \le 2^2 + 2^2 = 8$; on the other hand, $|x - y| \ge 1/4 + 1 + 1/4 = 1.5$. Therefore, $F(x,y) \le 8/1.5 < 6$. □

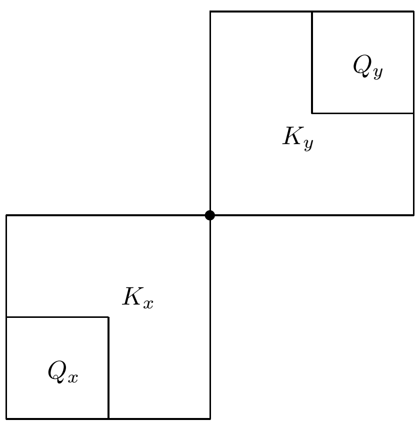

Fig. 7. The two squares

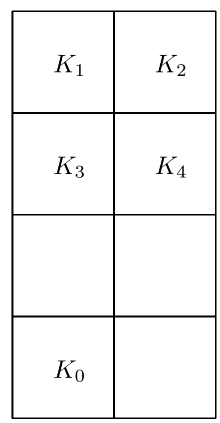

Fig. 8. The eight squares

Figure 8 shows eight unit squares. The following lemmas consider some possible positions of two points in these squares.

Lemma 6. *If $p(x)$ belongs to K_0 (in Fig. 8) and $p(y)$ belongs to K_2, K_3, or K_4, then $F(x,y) \le 6$.*

Proof. It is easy to see that, for any positive k, the "scaled" curve $kp(x/k^2)$ has the same square-to-linear dilation as the unit Peano–Hilbert curve $p(x)$. Therefore, taking the scale $k = 1/2^n$, we can assume that all of the squares K_i (in Fig. 8) are traversed in unit time. Moreover, the restriction of the curve to these squares is isometric to the unit Peano–Hilbert curve. In the proof of this lemma, we use p to denote the curve thus scaled.

We assume time to be oriented so that the curve starts in the square K_0 (to which $p(x)$ belongs) and, after some time, enters one of the squares (to which $p(y)$ belongs) listed in the statement of the lemma. Let a denote the distance from $p(x)$ to the edge through which the curve exits K_0, and let b be the distance from $p(y)$ to the edge through which the curve enters the square containing $p(y)$.

We denote the maximal possible squared distance between $p(x)$ and $p(y)$ by $D(a,b)$ and the minimal time required to go from $p(x)$ to $p(y)$ by $T(a,b)$. For the square-to-linear ratio, we have

$$F(x,y) \le \frac{D(a,b)}{T(a,b)}. \tag{4}$$

Let us consider the four possible positions of $p(y)$ in turn.

(1) Let $p(y) \in K_2$. If $a \ge 1/2$ and $b \ge 1/2$, then Corollary 1 implies

$$T(a,b) \ge 3 + \frac{1}{3}(a+b) \ge 3\frac{1}{3} = \frac{11}{3},$$

because the path from K_0 to K_2 passes through at least three unit squares. In turn, $D(a,b)$ has the obvious upper bound $4^2 + 2^2 = 20$. Therefore, in this case, we have

$$F(x,y) \le \frac{D(a,b)}{T(a,b)} \le \frac{60}{11} < 6.$$

If $a < 1/2$ and $b \ge 1/2$, then

$$T(a,b) \ge 3 + \frac{b}{3} \ge 3\frac{1}{6},$$

and $D(a,b)$ satisfies one of the following inequalities, depending on the edges through which the curve leaves K_0 and enters K_2:

$$D(a,b) \le \begin{cases} \left(\frac{1}{2} + b + 2\right)^2 + 2^2 & \text{if the exit is above and the entrance is below,} \\ \left(3 + \frac{1}{2}\right)^2 + (b+1)^2 & \text{if the exit is above and the entrance is on the left,} \\ (3+b)^2 + \left(1 + \frac{1}{2}\right)^2 & \text{if the exit is on the right and the entrance is below.} \end{cases}$$

The maximum value of $D(a,b)$ is attained at $b = 1$ and equal to $18\frac{1}{4}$. Thus, in the cases considered above, we obtain the following bound for the square-to-linear ratio:

$$F(x,y) \le \frac{18\frac{1}{4}}{3\frac{1}{6}} = \frac{219}{38} < 6.$$

In the case of the lateral entrance and exit, there is only the trivial bound $\le 4^2 + 2^2 = 20$ for $D(a,b)$, but the path passes through at least four unit squares, whence we have $T(a,b) > 4$ and $F(x,y) < 20/4 = 5 < 6$.

The case in which $a \ge 1/2$ and $b < 1/2$ is quite similar. It can even be formally reduced to the one considered above by inverting time and interchanging top with bottom and right with left.

Finally, suppose that $a < 1/2$ and $b < 1/2$. Then we have

$$D(a,b) \le \begin{cases} \left(\frac{1}{2} + \frac{1}{2} + 2\right)^2 + 2^2 & \text{if the exit is above and the entrance is below,} \\ \left(3 + \frac{1}{2}\right)^2 + \left(\frac{1}{2} + 1\right)^2 & \text{if the exit is above and the entrance is on the left,} \\ \left(3 + \frac{1}{2}\right)^2 + \left(1 + \frac{1}{2}\right)^2 & \text{if the exit is above and the entrance is below.} \end{cases}$$

The maximum value is $14\frac{1}{2}$, while $T(a,b)$ is, obviously, larger than 3. Therefore,

$$F(x,y) < 14\frac{1}{2}/3 < 6.$$

As above, in the case of the lateral entrance and exit we have $T(a,b) > 4$, and even the trivial estimate $D(a,b) < 4^2 + 2^2 = 20$ leads us to conclude that $F(x,y) < 20/4 = 5$ in this case.

Thus, we have completely considered the case of the second square.

(2) Let $p(y) \in K_3$. The structure of the Peano–Hilbert curve is such that, if the path from K_0 to K_3 is not straight, then it passes through at least three other squares. Therefore, the time required to go from the zeroth to the third square is at least 3, while the maximum squared distance between these squares does not exceed $3^2 + 1^2 = 10$. Hence we have $F(x,y) \le 10/3 < 4$ in this case.

In the case of a straight path, the exit is always above and the entrance is below; therefore, $D(a,b) \le (1+a+b)^2 + 1$. If $a > 1/2$ and $b > 1/2$, then $T(a,b) \ge 1 + (a+b)/3$. We obtain

$$F(x,y) \le \frac{(1+a+b)^2+1}{1+(a+b)/3}.$$

Since the derivative with respect to $a+b$ of the fraction on the right-hand side is positive for nonnegative $a, b \le 1$, it follows that this fraction attains its maximum at $a = b = 1$. This maximum is equal to 6.

If $a \ge 1/2 > b$, then $D(a,b) \le (a + 1/2 + 1)^2 + 1$ and $T(a,b) \ge 1 + a/3$. This implies

$$F(x,y) \le \frac{(a+1/2+1)^2+1}{1+a/3}.$$

The derivative with respect to a of the fraction on the right-hand side is positive; therefore, its maximum is attained at $a = 1$, and it is smaller than 6.

The case $a < 1/2 \le b$ is similar to the preceding one. Finally, if a and b are less than 1/2, then $D(a,b) \le 2^2 + 1 = 5$ and $T(a,b) > 1$, and hence $F(x,y) \le 5$.

(3) Suppose that $p(y) \in K_4$. In this case, the maximum squared distance between the points of the zeroth and fourth squares is equal to $2^2 + 3^2 = 13$; therefore, considering a path passing through more than two unit squares, we obtain $F(x,y) < 13/3$. If the path meets only two unit squares, then precisely one of the edges through which the path exits the zeroth square and enters the first one is lateral. Thus, it suffices to consider the case in which the exit is on the right and the entrance is from below. In this case, we have $D(a,b) \le (2+b)^2 + (1+a)^2$. If a or b is larger than 1/2, then $T(a,b) \ge 2 + 1/6$, and hence $F(x,y) \le 13/(2+1/6) = 6$. If $a, b < 1/2$, then $D(a,b) \le (2+1/2)^2 + (1+1/2)^2 = 8\frac{1}{2}$, and we obtain $F(x,y) < 5$, because $T(a,b) > 2$. □

Lemma 7. *If* $p(x) \in K_0$ (see Fig. 8) *and* $p(y) \in K_1$, *then* $F(x,y) \le 6$.

Proof. The maximum squared distance between points of K_0 and K_1 is equal to 17. Any non-straight path between them meets at least three squares. Therefore, $F(x,y) \le 17/3 < 6$. This means that it suffices to consider the case in which the traversal is as shown in Fig. 9.

The last third of the square K_0 (this square contains $p(x) = X$) and the first third of the square K_1 (containing $p(y) = Y$) are called the *nearest thirds*.

There are three different cases, depending on which thirds of the squares contain X and Y. In the first case, X and Y belong to the nearest thirds of the squares containing them (this case is shown in Fig. 10, left); in the second case, precisely one of the points X and Y belongs to the nearest third of its square (Fig. 10, middle); in the third case, both X and Y are outside the nearest thirds (Fig. 10, right).

The third case can be reduced to the second by choosing the point (X or Y) that is father from the left vertical boundary of the squares. Suppose that this point is X (see Fig. 10). Then we take a point Z in the nearest third of the first square at the same distance from X as Y; the time required to reach it is less than that required to reach Y, and hence $F(x,z) > F(x,y)$.

Consider the first case, in which the points X and Y belong to the nearest thirds of their squares (see Fig. 10). Let us denote the distance from X to the upper edge of its square by a and

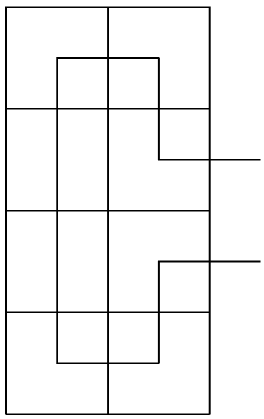

Fig. 9. The traversal

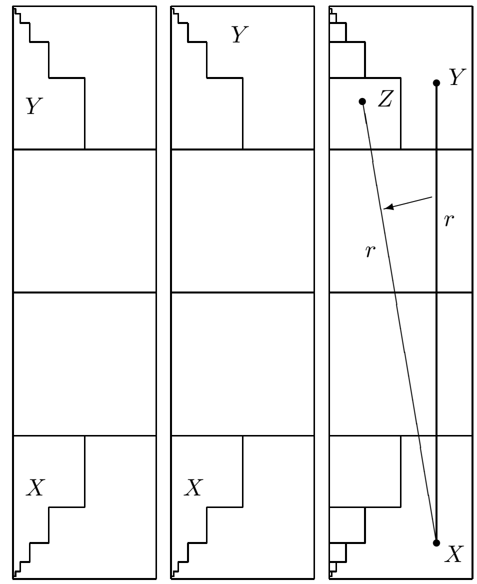

Fig. 10. Various domains

the distance from Y to the lower edge of its square by b. Suppose that $a < 1/2$ and $b < 1/2$. Then

$$|p(x) - p(y)|^2 \le 1^2 + (a + b + 2)^2 \le 1 + 3^2 = 10.$$

At the same time, $|x - y| \ge 2$, because the curve must pass through at least two unit squares while going from the zeroth square to the first. Therefore, we have $F(x, y) \le 5$ in this case.

Suppose that $a < 1/2$ and $b \ge 1/2$. Then, according to Corollary 1, we have

$$F(x, y) \le \frac{(1/2)^2 + (2 + 1/2 + b)^2}{2 + b/3}.$$

The function on the right-hand side increases and attains its maximum at $b = 1$; hence

$$F(x, y) \le \frac{75}{14} < 6.$$

Now, suppose that $a \ge 1/2$ and $b \ge 1/2$. Take positive integers k and l such that

$$1 - \frac{1}{2^{k-1}} \le a \le 1 - \frac{1}{2^k} \qquad \text{and} \qquad 1 - \frac{1}{2^{l-1}} \le b \le 1 - \frac{1}{2^l}. \tag{5}$$

Let $t(x)$ be the time needed to go from X to the exit from the zeroth square, and let $t(y)$ be the time needed to go from the entrance into the first square to Y. Then, by Lemma 3, we have

$$\frac{1}{3}\left(1 - \frac{1}{4^{k-1}}\right) \le t(x) \le \frac{1}{3}\left(1 - \frac{1}{4^k}\right) \qquad \text{and} \qquad \frac{1}{3}\left(1 - \frac{1}{4^{l-1}}\right) \le t(y) \le \frac{1}{3}\left(1 - \frac{1}{4^l}\right). \tag{6}$$

Let us prove that $F(x, y)$ is less than 6 for any k and l. To be definite, suppose that $k = l + s$, where $s \ge 0$ (the case $k < l$ is similar). The difference between the abscissas of the points $p(x)$ and $p(y)$ is at most $1/2^l$ by virtue of Lemma 3; therefore, we have

$$F(x, y) \le \frac{(2 + a + b)^2 + (1/2^l)^2}{2 + t(x) + t(y)}.$$

The inequality $F(x,y) < 6$ follows from

$$(2+a+b)^2 + \left(\frac{1}{2^l}\right)^2 < 12 + 6(t(x)+t(y)).$$

In turn, the last inequality is implied by the following relation, in which a and b are replaced by their upper bounds from (5) and $t(x)$ and $t(y)$ are replaced by their lower bounds from (6):

$$\left(4 - \frac{1}{2^k} - \frac{1}{2^l}\right)^2 + \left(\frac{1}{2^l}\right)^2 < 12 + 6\cdot\frac{1}{3}\left(1 - \frac{1}{4^{k-1}} + 1 - \frac{1}{4^{l-1}}\right).$$

Equivalent transformations reduce this inequality to the form

$$16 + \frac{1}{4^k} + \frac{2}{4^l} - 8\left(\frac{1}{2^k} + \frac{1}{2^l}\right) + \frac{2}{2^{k+l}} < 16 - \frac{2}{4^{k-1}} - \frac{2}{4^{l-1}},$$
$$\frac{9}{4^k} + \frac{10}{4^l} + \frac{2}{2^{k+l}} < 8\left(\frac{1}{2^k} + \frac{1}{2^l}\right),$$
$$9\frac{2^l}{2^k} + 10\frac{2^k}{2^l} + 2 < 8(2^k + 2^l),$$
$$9\cdot 2^{-s} + 10\cdot 2^s + 2 < 16(2^l + 2^{l+s}).$$

Since $l > 0$, it follows that $16\cdot 2^l > 2 + 9\cdot 2^{-s}$ and $16\cdot 2^{l+s} > 10\cdot 2^s$. Therefore, the very last inequality is true, and it implies the first one. Thus, for any $p(x)$ and $p(y)$ from the nearest thirds of their square fractions, we have $F(x,y) < 6$.

Now, consider the second case (see Fig. 10). Suppose that X belongs to the nearest third of its square and Y does not. (It is not necessary to consider the case where Y belongs to the nearest third because of the horizontal symmetry of the zeroth and first squares.)

Let us denote the distance from X to the upper edge of its square by a, the distance from X to the left edge of its square by b, and the distance from Y to the left edge of its square by c.

Suppose that $a < 1/2$. Then

$$|p(x) - p(y)|^2 \le 1^2 + (a+3)^2 \le 1 + \left(\frac{7}{2}\right)^2 = \frac{53}{4}.$$

At the same time, $|x-y| \ge 2 + 1/3$, because the curve must pass through at least two unit squares and the third containing Y while going from the zeroth square to the first one. Therefore, we have $F(x,y) < 6$ in this case.

Now, suppose that $a \ge 1/2$. By Lemma 3, we have

$$0 \le b \le \frac{1}{2^k}, \qquad a \le 1 - \frac{1}{2^{k-1}} \tag{7}$$

for some integer $k > 1$. Consider the upper square in more detail (see Fig. 11). We have the inequalities $1/2^{l+1} \le c \le 1/2^l$ for some l. Hence the point Y belongs to the first square of the lth-order fraction containing the upper left corner. This fraction consists of four subfractions of order $l+1$, which are traversed from the lower left one upward, to the right, and downward; the point Y is in the third or fourth subfraction. Therefore, the time $t(y)$ required to go from the upper left corner of the first square to Y is at least half the time required to traverse the lth-order fraction; i.e., we have

$$t(y) \ge \frac{1}{2}\cdot\frac{1}{4^l} = \frac{1}{2^{2l+1}}. \tag{8}$$

Let s denote the minimum of the numbers k and l. Then $F(x,y)$ is bounded above as follows:

$$F(x,y) \le \frac{(3+a)^2 + (1/2^s)^2}{t(x) + 7/3 + t(y)}, \tag{9}$$

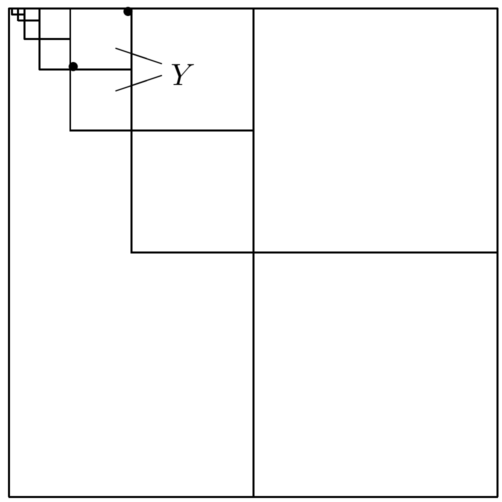

Fig. 11. The upper square

where $t(x)$ is the time required to go from X to the upper left corner of the zeroth square; according to Lemma 3, $t(x)$ satisfies the inequalities

$$\frac{1}{3}\left(1-\frac{1}{4^{k-1}}\right) \le t(x) \le \frac{1}{3}\left(1-\frac{1}{4^{k}}\right). \tag{10}$$

Replacing a in the numerator of the fraction (9) by its upper bound (7) and $t(x)$ and $t(y)$ in the denominator of this fraction by their lower bounds (10) and (8), respectively, we obtain

$$F(x,y) \le \frac{(3+1-1/2^{k-1})^2+(1/2^s)^2}{(1/3)(1-1/4^{k-1})+7/3+1/2^{2l+1}}. \tag{11}$$

Therefore, it suffices to show that the fraction on the left-hand side of (11) is less than 6 for any positive integers k and l, i.e., that

$$\left(4-\frac{2}{2^k}\right)^2+\frac{1}{4^s} \le 2\left(1-\frac{1}{4^{k-1}}\right)+14+\frac{3}{4^l}.$$

The left-hand side of this inequality is the numerator of the fraction (11), and its right-hand side is the sextupled denominator of this fraction. Opening parentheses, we obtain the equivalent inequality

$$16-\frac{16}{2^k}+\frac{4}{4^k}+\frac{1}{4^s} \le 2-\frac{8}{4^k}+14+\frac{3}{4^l}.$$

Reducing similar terms and transposing the negative summands, we transform it into the inequality

$$\frac{12}{4^k}+\frac{1}{4^s} \le \frac{16}{2^k}+\frac{3}{4^l},$$

in which, obviously, both terms on the left-hand side do not exceed the respective terms on the right-hand side. Therefore, the last inequality is true. This completes the proof of the inequality $F(x,y) \le 6$ for the second position of X and Y. We have considered all possible positions of the two points and proved the inequality $F(x,y) \le 6$ for all of them. □

Lemma 8. *The square-to-linear dilation factor of the Peano–Hilbert curve is at most* 6.

Proof. Take any two points with coordinates x and y in the interval. Their images in the square are $p(x)$ and $p(y)$.

Let n be the maximal positive integer for which x and y belong to neighboring (possibly, coinciding) nth-order fractions of the interval. For the $(n+1)$th-order fractions containing $p(x)$ and $p(y)$, the following cases are possible.

(1) The points x and y belong to the same nth-order fraction and $p(x)$ and $p(y)$ belong to neighboring $(n+1)$th-order fractions. In this case, we have

$$|p(x) - p(y)|^2 \le \left(\frac{2}{2^{n+1}}\right)^2 + \left(\frac{1}{2^{n+1}}\right)^2 = \frac{5}{4^{n+1}}.$$

By assumption, $|x - y| > 1/4^{n+1}$; hence $F(x, y) < 5$.

(2) The points x and y belong to the same nth-order fraction and $p(x)$ and $p(y)$ do not belong to neighboring fractions of order $n+1$. In this case, we denote the $(n+1)$th-order fraction containing $p(x)$ by K_x and the $(n+1)$th-order fraction containing $p(y)$ by K_y. In the case under consideration, these fractions must intersect in a corner. Therefore, Lemma 5 implies $F(x, y) < 6$.

It remains to consider the cases in which $p(x)$ and $p(y)$ belong to adjacent (along an edge) nth-order fractions, which we denote by K_x and K_y, respectively. Let $Q_x \subset K_x$ and $Q_y \subset K_y$ be the $(n+1)$th-order fractions containing $p(x)$ and $p(y)$, respectively.

(3) We have $Q_x \cap Q_y \ne \varnothing$. If Q_x and Q_y intersect in a corner, then the inequality $F(x, y) < 6$ follows from Lemma 5. If they intersect in an edge, then the path from Q_x to Q_y must meet at least two additional fractions of order $n+1$, because Q_x cannot immediately follow or precede Q_y because of the maximality of n. Therefore, $|x-y| \ge 2/4^{n+1}$ and, obviously, $|p(x)-p(y)|^2 \le 5/4^{n+1}$, whence $F(x, y) < 2.5$.

(4) We have $Q_x \cap K_y = \varnothing$ and Q_x is symmetric to Q_y with respect to the common edge of K_x and K_y, then the inequality $F(x, y) \le 6$ follows from Lemma 7.

(5) We have $Q_x \cap K_y = \varnothing$ and Q_x is not symmetric to Q_y with respect to the common edge of K_x and K_y, then $F(x, y) \le 6$ follows from Lemma 6.

(6) We have $Q_y \cap K_x = \varnothing$ and Q_x is not symmetric to Q_y with respect to the common edge of K_x and K_y. This case can be reduced to case (5) by changing the orientation of time. □

Theorem 1 follows from Lemmas 2 and 8.

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K. E. Bauman
M. V. Lomonosov Moscow State University