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One-Side Peano Curves of Fractal Genus 9

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Abstract—This paper completes the analysis (begun by E.V. Shchepin and the author in 2008) of regular Peano curves of genus 9 in search of a curve with the minimum square-to-linear ratio. One-side regular Peano curves of genus 9 are considered, and, among these curves, a class of minimal curves with a square-to-linear ratio of $5\frac{2}{3}$ is singled out. A new language to describe curves is introduced which significantly simplifies the coding of these curves.

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1. INTRODUCTION

In the 19th century, the Italian mathematician Giuseppe Peano learned how to construct continuous surjective maps of a closed interval onto a square, which were later called Peano curves. Since then, a variety of methods have been developed for constructing similar curves. In [6] Sagan describes the most popular of these methods. In [2, 3, 8, 9], a variety of applications of Peano curves are demonstrated.

An important characteristic of Peano curves is the so-called square-to-linear ratio. For a pair $p(t), p(\tau)$ of points of a Peano curve $p\colon [0,1] \to [0,1]^2$, the quantity

$$\frac{|p(t) - p(\tau)|^2}{|t - \tau|}$$

is called the *square-to-linear ratio* (SLR) of the curve p on this pair. The supremum of the SLRs over all possible pairs of different points of the curve is called the square-to-linear ratio of the curve. For applications, of interest are the curves with the least possible SLR (see [2]).

2. KNOWN ESTIMATES

In the same way that an ordinary curve is parameterized by its length, a Peano curve is naturally parameterized by its area. Namely, a Peano curve $p(t)$ is said to be *parameterized by area* if the area of the image of any interval is equal to the length of this interval.

All the Peano curves considered below are assumed to be parameterized by area.

A closed interval contained in the domain of definition of a Peano curve is called a *fractal period* of this curve if the restriction to this interval is similar to the whole curve.

The restriction of a curve to its fractal period is called a *fraction* (or section) of this curve.

In [2], Shchepin introduced the concept of a *regular fractal Peano curve* as a mapping of a closed interval onto a square whose domain of definition can be partitioned into several equal intervals (fractal periods) such that the restriction of the curve to each of them is similar to the whole curve.

Theorem 3 in [2] erroneously states that the beginning and end of any regular fractal square Peano curve coincide with the vertices of the image square. The proof given in [2] is valid only when the operation of time reversal is not used at the second step of the construction of the curve. When the image is a square, the class of such curves coincides with the class defined by Wunderlich [7].

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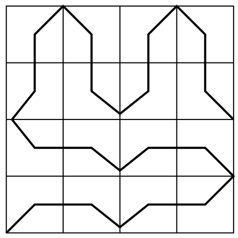

Fig. 1. Counterexample to Theorem 3 from [2].

Figure 1 presents a counterexample to Theorem 3 from [2]; namely, the figure shows the second step of constructing a curve of fractal genus 16 that has its starting point at a vertex of a square and the endpoint at the midpoint of a side.

In [8], for a wide class of curves including regular fractal Peano ones, Haverkort and van Walderveen proved that the maximum of the SLR is not less than 4.

Theorems 4 and 5 in [2] state that the SLR of a regular fractal Peano curve with endpoints at the vertices of a square cannot be less than 5. In particular, this is valid for Wunderlich's square curves.

On the other hand, it is proved in [5] that the classical Peano–Hilbert curve has the maximum SLR equal to 6.

A regular Peano curve is said to be *one-side* if its endpoints lie at the vertices of the same side of a square. If they lie at opposite vertices of the square, then the curve is said to be *diagonal*.

In [4], Shchepin and the author considered the class of regular diagonal Peano curves of fractal genus 9 and found a unique, up to similarity, curve whose SLR is less than 6. The SLR for this curve turned out to be $5\frac{2}{3}$.

The following question remains open: Does there exist a regular square Peano curve whose SLR is less than $5\frac{2}{3}$?

In the present study, we complete the analysis of regular Peano curves of fractal genus 9 in search of a curve with the least SLR.

3. ARRANGEMENT OF THE ENDPOINTS OF PEANO CURVES

Lemma 1. *At least one endpoint of a regular Peano curve lies at a vertex of the square.*

Proof (these arguments appeared in Shchepin's paper [2] in the proof of Theorem 3)**.** Suppose that neither endpoint of the curve lies at the vertices of the square. Hence, they may lie either on the same, on adjacent, or on opposite sides of the square.

The case of the same side is impossible because the entry and exist in the first fraction would be on a part of the boundary of the square that has no intersection with other fractions.

The case of opposite sides is also impossible because the curve (more precisely, the polygonal path with vertices at the centers of fractions) would not be able to make a turn; hence, it would be impossible for the curve to "visit" all fractions.

Suppose that the endpoints of the curve lie on adjacent sides. Let us number the fractions of each step in the order they are traversed by the curve. Then the intersection of fractions with numbers differing by 2 is nonempty. Consider a corner fraction at the third step of the construction with at least two fractions passed before it and two fractions passed after it. This fraction must have a nonempty intersection with the two predecessors and with the two direct successors, i.e., with four other fractions. But the number of neighbors of a corner fraction is only three. A contradiction. $\square$

Theorem 1. *In any regular Peano curve, either both endpoints are at the vertices of the square, or one of them is at a vertex while the other is at the midpoint of one of two opposite sides.*

Proof. By Lemma 1, either the starting or the end point lies at a vertex of a square. Suppose that the starting point lies at the lower left vertex of the square; otherwise we will consider the curve after the time reversal and/or appropriate rotation.

Suppose that the end point lies neither at a vertex nor at the midpoint of a side of the square.

The case when the end point lies on the same side as the starting point is impossible because then the entry and exit in the first fraction would lie on a part of the boundary of the square that has no intersection with other fractions.

Suppose that the end point lies on a side opposite to the starting point, for example, on the right vertical side of the square, and does not coincide with the midpoint of the side; i.e., it is located at a distance of x from the lower boundary of the square, with $x \notin \{0, \frac{a}{2}, a\}$, where a is the side length of the square. In this case, at the second step of construction, any fraction oriented in the same way as the whole curve may only be followed by a fraction that is mirror symmetric to it with respect to their common boundary. Since the first fraction, just as the whole curve, has its beginning at a vertex, the end of the second fraction also lies at a partition vertex. Hence, all odd (in the order of traversal) fractions begin at partition vertices, while even fractions end at vertices.

Introduce coordinates with the origin at the lower left corner of the square and the axes directed along the sides of the square; as a unit, we take the side length of a fraction at the second step of the construction of the curve.

Note that the coordinates of the end of the second fraction can be either $(2,0)$ or $(0,2)$ and that the coordinates of any end or beginning of a fraction are even numbers if this end/beginning lies at a partition vertex.

Suppose that the curve has even fractal genus. Then the upper left fraction has its starting or end point at the upper left vertex of the square. This vertex cannot be the endpoint of the whole curve, neither can this fraction have two neighbors in the order of traversal. A contradiction.

Suppose that the curve has odd fractal genus. Then the upper right fraction starts at its lower left corner and ends on a side of the large square, i.e., on a side of the fraction that has no intersections with other fractions. Hence, such a fraction may only be the final one. The fraction is similar to the whole curve, but the corresponding similarity mapping has only one fixed point at the corner of the square; hence, the endpoint of the curve lies at the upper right corner of the square, which contradicts the initial assumption. $\square$

Lemma 2. *Any regular Peano curve of genus* 9 *has its endpoints at the vertices of the square.*

Proof. By Theorem 1, it suffices to eliminate the case when the endpoint of a curve lies at the midpoint of a side.

Suppose that a regular Peano curve of fractal genus 9 has its starting point at the lower left corner, and the end point at the midpoint of the right side. The entry to the last fraction is at a partition vertex, because it is the ninth (i.e., an odd) fraction in the order of traversal. In this case, the upper right fraction cannot have two neighbors (passed directly before or after it). A contradiction. $\square$

Lemma 2 implies that regular Peano curves of genus 9 may be either diagonal or one-side; hence, based on Theorems 4 and 5 from [2], we can argue that the SLR of regular Peano curves of genus 9 is not less than 5.

In [4], among all diagonal curves of genus 9, a minimal curve with the SLR equal to $5\frac{2}{3}$ was found. In the present study, we consider one-side curves and thus complete the analysis of curves of fractal genus 9.

4. CODE OF THE SECOND STEP

4.1. Vertex code. We consider a plane equipped with a complex structure. We assume that the real axis is horizontal and directed from left to right and the imaginary axis is directed from bottom to top.

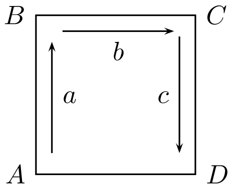

Fig. 2. The first step of a one-side Peano curve.

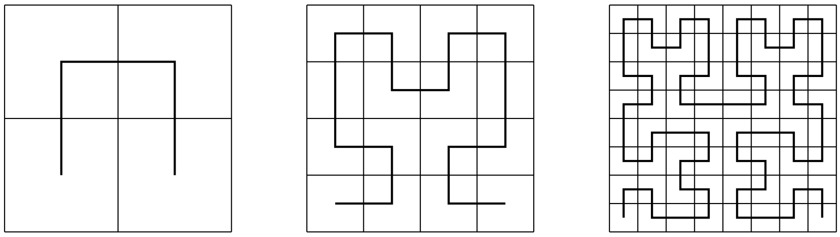

Fig. 3. Three steps of constructing the Peano–Hilbert curve.

We regard a curve as the trajectory of a continuously moving point in a unit time interval. Then a *corner moment* is the time from the start to the instant when the point reaches a corner of the square.

To code the order of vertices passed by a moving point, we will use a vertex code which is a sequence of symbols from $\{\pm i, \pm 1\}$ enclosed in brackets. The order of these symbols indicates the direction of motion from a current vertex at some step of constructing a curve to the next vertex. For example, the code $[i+1-i]$ corresponds to a sequence in which the second vertex is located above the first, the third is to the right of the second, and the fourth, under the third (Fig. 2).

Figure 2 shows the vertex code of the first step of constructing all one-side Peano curves that start at the lower left corner and end at the lower right corner, with the upper vertices traversed from left to right. In particular, such is the first step of the Peano–Hilbert curve (Fig. 3). The vertex code of the second step of constructing this curve is

$$[1+i-1+i+1-i+i+1-i-1-i+1].$$

We will consider the symbols of a vertex code to be complex numbers; therefore, the code can be (termwise) multiplied by a complex number. For example, $i[i+1-i]=[-1+i+1]$. Complex conjugation is also applied termwise to the codes. Thus, $\overline{[i+1-i]}=[-i+1+i]$. Using the notation adopted and denoting the code $[i+1-i]$ by a single letter d, we can represent the above code row of the second subdivision as follows:

$$[i\overline{d}+d+d-i\overline{d}\,].$$

This formula is universal; it allows one to obtain the code of the $(n+1)$th subdivision by substituting the code of the nth subdivision for d. Thus, we have found a recurrent equation for the vertex code of the Peano–Hilbert curve:

$$d_{n+1}=[i\overline{d_n}+d_n+d_n-i\overline{d_n}], \qquad d_1=[i+1-i].$$

Introduce the operation of time reversal and denote it by d^{-1}. In Fig. 2, the lowercase Roman letters stand for the time taken for the curve to get from one vertex to another; i.e., each arrow is assigned its time. In this case, the vertex code can be written as $[i|a+1|b-i|c]$. The time reversal consists in changing the order of traversal of the square to the mirror symmetric one and changing the time sequence on the sides of the square to the opposite; i.e., the above code takes the form $[i|c+1|b-i|a]$.

If the vertex code of a one-side curve at the first step is $[i+1-i]$, then the operation of time reversal acts on the vertex code of some step of construction as follows: first, the order of elements is reversed, and then the conjugate of the vertex code is taken.

Note that a curve with the code of the first step equal to $[i+1-i]$ lies above the real axis and to the right of the imaginary axis. When we reverse the order of numbers in the code, the vertex code of the first subdivision turns into $[-i+1+i]$, which lies below the real axis. To bring the curve back to the original place, we apply the operation of conjugation.

Under time reversal, the vertex code of a one-side curve expressed in terms of d is transformed as follows: the order of elements is reversed, the coefficients of d are replaced by their conjugates, and all d are changed to d^{-1} (with $(d^{-1})^{-1}=d$).

To illustrate the procedure, we demonstrate it as applied to an asymmetric one-side curve:

$$\big[d+i\overline{d}+i\overline{d}-d+d+d+d-i\overline{d}-i\overline{d}\,\big]^{-1}=\big[i\overline{d^{-1}}+i\overline{d^{-1}}+d^{-1}+d^{-1}+d^{-1}-d^{-1}-i\overline{d^{-1}}-i\overline{d^{-1}}+d^{-1}\big].$$

Now, to define a curve, it suffices to present its vertex code at the first step and a recurrent equation for the vertex code, i.e., an expression for the code of the $(n+1)$th step of constructing the curve in terms of the nth-step code.

4.2. Junctions. A kth-order junction is the restriction of a curve to a pair of adjacent fractions of the kth step of construction.

The vertex code of a junction is a pair of codes of the whole curve with appropriate orientations. For example, the code $[i\overline{d}+d]$ corresponds to the junction of the first and second fractions of the Peano–Hilbert curve at the first division step.

To the code of a junction, one can apply the same operations as to the code of the whole curve.

Two junctions are said to be *equivalent* if the vertex code of one junction can be obtained from the vertex code of the other by the operations of conjugation, multiplication by i, and time reversal. A junction is said to be *primitive* if the first part of its code is $d^{\pm 1}$. For example, after conjugation and double multiplication by i, the junction $[-\overline{d}-id]$ turns into the primitive junction $[d-i\overline{d}]$.

For any junction, there exists an equivalent primitive junction.

4.3. Derived junctions. A junction of fractions of the kth subdivision is called a *derived junction* of a junction of fractions of a coarser subdivision if its first and second fractions lie in the first and second fractions of the coarser junction, respectively.

Using the code of a junction and the recurrent equation of a curve, one can obtain the code of the derived junction of the next step. To this end, one should substitute d_k for d into the code of the junction, express d_k in terms of d_{k-1} with the recurrent equation, take the last monomial with d_{k-1} from the first part and the first monomial from the second part, and finally substitute d back for d_{k-1}. The pair obtained is the code of the derived junction.

As an example, let us calculate the code of the derived junction of the first and second fractions in the second step of the construction of the Peano–Hilbert curve. The junction of the first and second fractions is coded by $[i\overline{d}+d]$. Replacing d_k with its expression in terms of d_{k-1}, we obtain

$$\begin{aligned}[i\overline{d_k}+d_k]&=\big[\overline{i\{i\overline{d_{k-1}}+d_{k-1}+d_{k-1}-i\overline{d_{k-1}}\}}+\{i\overline{d_{k-1}}+d_{k-1}+d_{k-1}-i\overline{d_{k-1}}\}\big]\\&=\big[\{d_{k-1}+i\overline{d_{k-1}}+i\overline{d_{k-1}}-d_{k-1}\}+\{i\overline{d_{k-1}}+d_{k-1}+d_{k-1}-i\overline{d_{k-1}}\}\big].\end{aligned}$$

From the first part, we take the last monomial with d_{k-1}, and from the second part, the first monomial; replace d_{k-1} by d; and obtain the code $[-d+i\overline{d}]$ of the derived junction.

Following Shchepin [4], we define the *depth* of each junction of a curve as the least k for which this junction is similar to the junction of a pair of fractions of the kth subdivision.

For a regular Peano curve, we define its *depth* as the maximum of the depths of its junctions.

5. ONE-SIDE PEANO CURVES OF FRACTAL GENUS 9 WITH DIAGONAL TRANSITION

Everywhere below, we consider regular Peano curves of fractal genus 9.

There are plenty of one-side curves of fractal genus 9, but we are interested only in those whose SLR is not greater than $5\frac{2}{3}$.

We will say that a curve has a diagonal transition if there exist adjacent fractal periods that are mapped to squares intersecting at a single vertex.

Theorem 2. *The SLR of Peano curves of fractal genus* 9 *that have a diagonal transition is strictly greater than* $5\frac{2}{3}$.

Proof. Suppose that, at the first step of construction, a curve has a diagonal transition. Let α and $1-\beta$ be the first and second corner moments, respectively; i.e., α is the time from the entry into the square to the next (second) vertex divided by the time spent in the whole square, while β is the time from passing the third vertex to exiting from the square divided by the time spent in the whole square. Assume that $\alpha \le \beta$ (otherwise we reverse time $t \to 1-t$, thus interchanging α and β and satisfying the inequality).

The fractions involved in a diagonal transition may be oriented with respect to each other in two ways (Fig. 4). In any case, we consider the points $A_1 = p(x_1)$ and $A_2 = p(x_2)$. We have

$$F(x_1, x_2) = \frac{(p(x_1) - p(x_2))^2}{|x_1 - x_2|} \ge \frac{8}{2-2\alpha} = \frac{4}{1-\alpha}.$$

The condition $F(x_1, x_2) \le 5\frac{2}{3}$ implies $\frac{4}{1-\alpha} \le 5\frac{2}{3} \Rightarrow \alpha \le \frac{5}{17} < \frac{1}{3}$. The latter condition may only hold when the curve at the first step of construction has a direct path from the entry to the first corner square. Hence, only three curves from the class of Peano curves of fractal genus 9 with diagonal transition remain of interest to us (Fig. 5).

Let $t(AB) = |p^{-1}(A) - p^{-1}(B)| \cdot 9^n$ be the relative time between points A and B that belong to the same fraction of the nth step of constructing the curve.

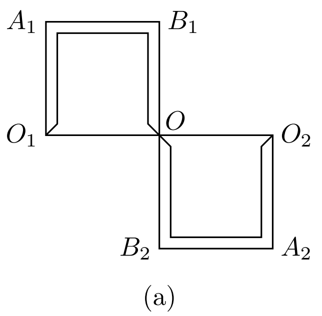

Fig. 4. Variants of diagonal transitions.

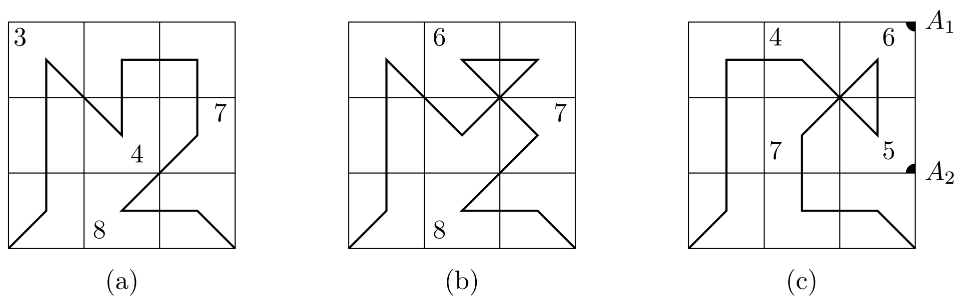

Fig. 5. Three curves with diagonal transition.

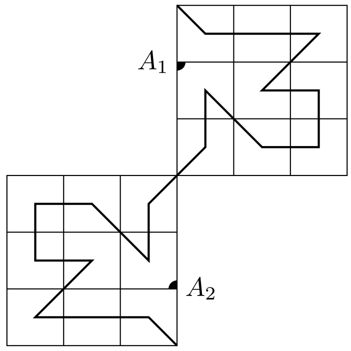

Fig. 6. Diagonal transition between the seventh and eighth fractions.

Lemma 3. *In the class of one-side curves of genus* 9 *with diagonal transition of the first type* (*Fig.* 4a), *the condition* $F(x,y) \le 5\frac{2}{3}$ *may hold at the vertices of these fractions only if* $t(O_1A_1) < t(OB_1)$ *and* $t(O_2A_2) < t(OB_2)$.

Proof. If $F(A_1,A_2) \le 5\frac{2}{3}$, then

$$\frac{8}{2 - t(O_1A_1) - t(O_2A_2)} \le 5\frac{2}{3} \qquad \Rightarrow \qquad t(O_1A_1) + t(O_2A_2) \le \frac{10}{17}.$$

On the other hand, from the inequality $F(B_1,B_2) \le 5\frac{2}{3}$ we obtain

$$\frac{4}{t(OB_1) + t(OB_2)} \le 5\frac{2}{3} \qquad \Rightarrow \qquad t(OB_1) + t(OB_2) \ge \frac{12}{17}.$$

Notice that $t(OB_1)$ and $t(O_1A_1)$ ($t(OB_2)$ and $t(O_2A_2)$) equal either α or β, and they cannot coincide because they are different corner moments in the same square. Taking into account the inequalities obtained, we find that, under such a diagonal transition, the orientations of the fractions are defined uniquely; i.e., $t(OB_1) = t(OB_2) = \beta$ and $t(O_1A_1) = t(O_2A_2) = \alpha$. $\square$

Now, to complete the proof of Theorem 2, we consider the three remaining curves with a direct path from the entry to the first corner moment (Fig. 5). The SLR of other curves of fractal genus 9 with diagonal transition is certainly greater than $5\frac{2}{3}$.

The first curve (Fig. 5a). Suppose that the SLR of this curve is not greater than $5\frac{2}{3}$. Then, by Lemma 3, the orientations of the third, fourth, seventh, and eighth fractions are defined uniquely because they are involved in diagonal transitions of the first type (Fig. 4a). Note that $\alpha = \frac{2}{9} + \frac{\alpha}{9}$ $\Rightarrow \alpha = \frac{1}{4}$ in this case. Then we consider the diagonal transition between the seventh and eighth fractions in greater detail, at the third step of construction (Fig. 6). Let us calculate the SLR at the marked points:

$$F(A_1,A_2) = \frac{4^2}{2+2\alpha} = \frac{16}{5/2} = \frac{32}{5} > 5\frac{2}{3}.$$

Hence, the SLR of this curve is greater than $5\frac{2}{3}$.

The second curve (Fig. 5b). Suppose that the SLR of this curve is not greater than $5\frac{2}{3}$. Since the diagonal transition between the sixth and seventh fractions is of the first type, Lemma 3 defines their orientations uniquely. However, on the other hand, since the diagonal transition between the seventh and eighth fractions is also of the first type, the orientation of the seventh fraction should be opposite to the one just defined. We obtain a contradiction. Hence, the SLR of this curve is strictly greater than $5\frac{2}{3}$.

The third curve (Fig. 5c). Suppose that the SLR of this curve is not greater than $5\frac{2}{3}$. Since the diagonal transition between the fourth and fifth fractions is of the first type (Fig. 4a), Lemma 3 uniquely defines the orientations of these fractions. Consider the junction of the fifth and sixth fractions in more detail.

Recall that $\alpha \le \frac{5}{17}$. If the sixth fraction is oriented in time like the whole curve, then the SLR at the points A_1 and A_2 is estimated as follows:

$$F(A_1, A_2) = \frac{2^2}{2\alpha} = \frac{2}{\alpha} > 5\frac{2}{3}.$$

Otherwise we calculate the second corner moment, $\beta = \frac{4-\beta}{9} \Rightarrow \beta = \frac{2}{5}$, and estimate the SLR at the indicated points (Fig. 5c):

$$F(A_1, A_2) = \frac{2^2}{\alpha + \beta} \ge \frac{4}{\frac{2}{5} + \frac{5}{17}} = \frac{340}{59} > 5\frac{2}{3}.$$

Hence, the SLR of this curve is also strictly greater than $5\frac{2}{3}$. $\square$

6. ONE-SIDE PEANO CURVES OF FRACTAL GENUS 9 WITHOUT DIAGONAL TRANSITION

6.1. Uniqueness of the first step.

Lemma 4. *There exists a unique, up to isometry, first step of the construction of one-side curves of genus* 9 *without diagonal transition.*

Proof. For convenience, we enumerate the squares line by line from top to bottom. Without loss of generality, assume that the curve starts at the lower left corner (i.e., in the seventh square) and ends at the lower right corner (i.e., in the ninth square). We have to construct a path passing through all nine squares so that any two consecutive squares share a side.

After the first square, the curve has two alternative ways: either to the right or upward. In the first case, notice that only the square with number five can become the third square. Next, the curve may go only to the left, then upward, and so on. Hence the traversal order $(7, 8, 5, 4, 1, 2, 3, 6, 9)$ is defined uniquely. In the second case similar arguments lead to the same curve, but mirror reflected. $\square$

It follows from Lemma 4 that all one-side curves of fractal genus 9 without diagonal transition differ from each other only by the orientation of fractions at the second step of construction. Denote the first step of such curves by $d = [i+1-i]$. Fix the traversal of fractions at the first step according to Fig. 7. Then the recurrent equation of all such curves is given by

$$d_{n+1} = \left[d_n^{\pm 1} + i\overline{d_n^{\pm 1}} + i\overline{d_n^{\pm 1}} - d_n^{\pm 1} + d_n^{\pm 1} + d_n^{\pm 1} + d_n^{\pm 1} - i\overline{d_n^{\pm 1}} - i\overline{d_n^{\pm 1}}\right].$$

6.2. The set of minimal one-side curves of genus 9.

Denote by M the set consisting of four curves whose first step is d and the recurrent equation is

$$d_{n+1} = \left[d_n + i\overline{d_n} + i\overline{d_n^{-1}} - d_n^{-1} + d_n + d_n^{\pm 1} + d_n^{-1} - i\overline{d_n^{\pm 1}} - i\overline{d_n^{-1}}\right].$$

Notice that these curves are very similar to each other. The difference lies in the orientations of the sixth and eighth (in the order of traversal) fractions.

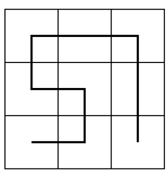

Fig. 7. Traversal of one-side curves of genus 9 without diagonal transition.

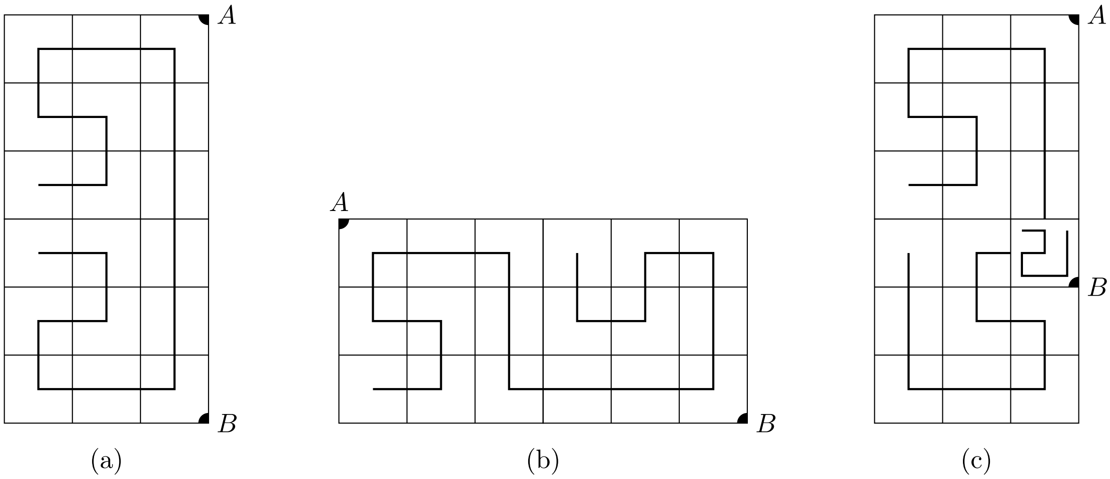

Fig. 8. The junctions (a) $[d-d^{-1}]$, (b) $[d+i\overline{d^{-1}}]$, and (c) $[d-d]$.

Theorem 3. *If a one-side curve of fractal genus* 9 *without diagonal transition has the SLR not greater than* $5\frac{2}{3}$, *then it belongs to the set* M.

The proof of the theorem is based on a series of lemmas that restrict the freedom in choosing the orientations of all fractions of the second step, except for the sixth and eighth.

For convenience, henceforth we will refer to regular one-side Peano curves of genus 9 without diagonal transition with SLR not greater than $5\frac{2}{3}$ as a set of minimal curves.

Lemma 5. *The corner moments of a curve with the first step* d *can take the following values*:

$$\alpha \in \left\{\frac{17}{36}, \frac{19}{40}, \frac{1}{2}\right\}, \qquad \beta \in \left\{\frac{1}{4}, \frac{11}{40}, \frac{5}{18}\right\}.$$

Proof. Depending on the orientations of the corner fractions, at the second stage of construction we obtain four systems for α and β:

$$\alpha = \frac{4}{9} + \frac{\beta}{9}, \qquad \beta = \frac{2}{9} + \frac{\alpha}{9} \qquad \Rightarrow \qquad \alpha = \frac{19}{40}, \quad \beta = \frac{11}{40}; \tag{6.1}$$

$$\alpha = \frac{4}{9} + \frac{\alpha}{9}, \qquad \beta = \frac{2}{9} + \frac{\beta}{9} \qquad \Rightarrow \qquad \alpha = \frac{1}{2}, \quad \beta = \frac{1}{4}; \tag{6.2}$$

$$\alpha = \frac{4}{9} + \frac{\alpha}{9}, \qquad \beta = \frac{2}{9} + \frac{\alpha}{9} \qquad \Rightarrow \qquad \alpha = \frac{1}{2}, \quad \beta = \frac{5}{18}; \tag{6.3}$$

$$\alpha = \frac{4}{9} + \frac{\beta}{9}, \qquad \beta = \frac{2}{9} + \frac{\beta}{9} \qquad \Rightarrow \qquad \alpha = \frac{17}{36}, \quad \beta = \frac{1}{4}. \tag{6.4}$$

Lemma 6. *If the vertex code of a curve contains a junction equivalent to the junction* $[d-d^{-1}]$, *then the SLR of the curve is greater than* $5\frac{2}{3}$.

Proof. Consider the SLR between the points A and B in Fig. 8a:

$$F(A,B) \geq \frac{2^2}{\frac{5}{18}\cdot 2} = \frac{36}{5} > 5\frac{2}{3}. \quad \square$$

Lemma 7. *If the vertex code of a curve has a junction equivalent to the junction* $[d+i\overline{d^{-1}}]$, *then the SLR of the curve is greater than* $5\frac{2}{3}$.

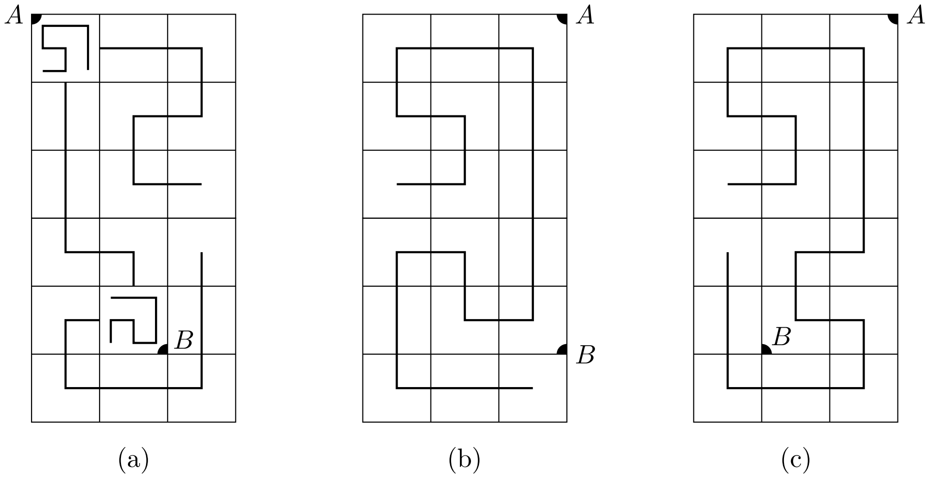

Fig. 9. (a) Junction of the fourth and fifth fractions, and junctions (b) $[d - i\overline{d}]$ and (c) $[d - d]$.

Proof. Consider the SLR between the points A and B in Fig. 8b:

$$F(A,B) \geq \frac{2^2+1^2}{\frac{5}{18}+\frac{19}{36}} = \frac{180}{29} > 5\frac{2}{3}. \quad \square$$

Lemma 8. *If the vertex code of a curve has a junction equivalent to $[d - d]$ and the first fraction of the second step has the code d^{-1}, then the SLR of the curve is greater than $5\frac{2}{3}$.*

Proof. Consider the SLR between the points A and B in Fig. 8c:

$$F(A,B) \geq \frac{\frac{16}{9}}{\frac{5}{18}\cdot\frac{10}{9}} = \frac{144}{25} > 5\frac{2}{3}. \quad \square$$

Lemma 9. *The code of the fifth fraction of the second step of constructing a minimal curve is d.*

Proof. Suppose that the code of the fifth fraction is d^{-1}. Then, by Lemma 6, the fourth fraction has the code $-d^{-1}$; hence, by Lemma 7, the code of the third fraction is $i\overline{d^{-1}}$.

Consider the SLR between the points A and B in Fig. 9a:

$$F(A,B) \geq \frac{2^2+5^2}{\alpha+2+2+1-\alpha} = \frac{29}{5} > 5\frac{2}{3}. \quad \square$$

Corollary 1. *The first corner moment of minimal one-side Peano curves of genus 9 is $\frac{1}{2}$.*

Lemma 10. *The code of the first fraction of minimal curves is d.*

Proof. Suppose that the code of the first fraction is d^{-1}. Then, at the junction of the fourth and fifth fractions, depending on the orientation of the fourth fraction, the inequality $F(A,B) \leq 5\frac{2}{3}$ is violated either by Lemma 6 or by Lemma 8. A contradiction. $\square$

Lemma 11. *The code of the second fraction of minimal curves is $i\overline{d}$.*

Proof. The claim follows from Lemmas 10 and 7. $\square$

Lemma 12. *The code of a minimal curve cannot have a junction equivalent to $[d - i\overline{d}]$.*

Proof. Consider the SLR between the points A and B in Fig. 9b. Notice that Lemma 11 determines the orientation of the second fraction. Hence,

$$F(A,B) \geq \frac{\frac{25}{9}}{\frac{5}{18}+\frac{1}{9}\cdot\frac{3}{2}} = \frac{25}{4} > 5\frac{2}{3}. \quad \square$$

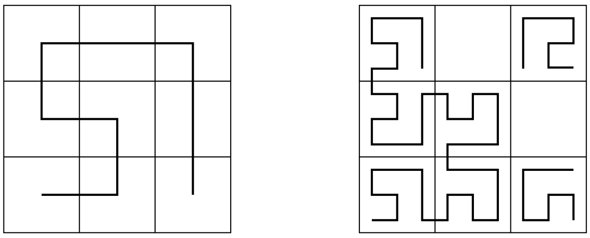

Fig. 10. The first steps of constructing minimal one-side curves of genus 9.

Lemma 13. *The code of the third fraction of minimal curves is* $i\overline{d^{-1}}$.

Proof. Suppose that the code of the third fraction is $i\overline{d}$. Then the junction of the third and fourth fractions has either the form $[i\overline{d} - d]$ or the form $[i\overline{d} - d^{-1}]$. The former is equivalent to $[d - i\overline{d}]$ and is impossible by Lemma 12, while the latter is equivalent to $[d + i\overline{d^{-1}}]$ (use the time reversal) and is impossible in view of Lemma 7. A contradiction. $\square$

Lemma 14. *The code of a minimal curve cannot have a junction equivalent to* $[d - d]$.

Proof. Consider the SLR between the points A and B in Fig. 9c. Note that Lemma 13 determines the orientation of the third fraction. Hence,

$$F(A,B) \geq \frac{\frac{25}{9} + \frac{4}{9}}{\frac{1}{9} \cdot \frac{5}{2} + \frac{5}{18}} = \frac{29}{5} > 5\frac{2}{3}. \quad \square$$

Corollary 2. *The code of the fourth fraction of minimal curves is* $-d^{-1}$.

Proof. By Lemma 14, the junction of the fourth and fifth fractions is defined uniquely for minimal curves. This implies the assertion of the corollary. $\square$

Lemma 15. *The code of the ninth fraction of minimal curves is* $-i\overline{d^{-1}}$.

Proof. Suppose that the code of the ninth fraction is $-i\overline{d}$. In this case, since the junction of the second and third fractions has the form $[i\overline{d} + i\overline{d^{-1}}]$, its derived junction is $[-d + d^{-1}]$. The latter is equivalent to the junction $[d - d^{-1}]$, which is impossible in minimal curves in view of Lemma 6. A contradiction. $\square$

Lemma 16. *The code of the seventh fraction of minimal curves is* d^{-1}.

Proof. Suppose that the code of the seventh fraction is d. In this case, the junction of the seventh and eighth fractions has either the form $[d - i\overline{d}]$ or the form $[d - i\overline{d^{-1}}]$, depending on the orientation of the eighth fraction. However, the former is impossible in minimal curves by Lemma 12, while the latter is equivalent to the junction $[d + i\overline{d^{-1}}]$ and hence is also impossible by Lemma 7. $\square$

Proof of Theorem 3. So, the class of curves satisfying Lemmas 9–11, 13, 15, and 16 and Corollary 2 is restricted to the four curves (Fig. 10)

$$\big[d + i\overline{d} + i\overline{d^{-1}} - d^{-1} + d + d^{\pm 1} + d^{-1} - i\overline{d^{\pm 1}} - i\overline{d^{-1}}\big].$$

6.3. Square-to-linear ratio of the curves in the set M**.**

Lemma 17. *The depth of any curve in the set* M *is either* 1 *or* 2.

Proof. The set M consists of four curves that differ by the orientations of the sixth and eighth (in the order of traversal) fractions.

Let us list the junctions of the first order and reduce them to primitive ones.

Each of the four curves has the following junctions:

$$[d + i\overline{d}], \quad [i\overline{d} + i\overline{d^{-1}}] \to [d + d^{-1}], \quad [i\overline{d^{-1}} - d^{-1}] \to [d + i\overline{d}], \quad [-d^{-1} + d] \to [d^{-1} - d].$$

If the code of the sixth fraction is d, then the junctions $[d+d]$ and $[d+d^{-1}]$ arise, whereas if this code is d^{-1}, then $[d+d^{-1}]$ and $[d^{-1}+d^{-1}] \to [d+d]$ arise. Hence, the set of junctions is independent of the orientation of the sixth fraction.

If the code of the eighth fraction is $-i\overline{d^{-1}}$, then the junctions $[d^{-1}-i\overline{d^{-1}}] \to [d+i\overline{d}]$ and $[-i\overline{d^{-1}}-i\overline{d^{-1}}] \to [d+d]$ arise, while if the code of the eighth fraction is $-i\overline{d}$, then we have $[d^{-1}-i\overline{d}]$ and $[-i\overline{d}-i\overline{d^{-1}}] \to [d+d^{-1}]$.

Thus, any curve in the set M contains the following junctions at the first step of construction: $[d+i\overline{d}]$, $[d+d^{-1}]$, $[d^{-1}-d]$, and $[d+d]$. Only two curves with the code of the eighth fraction equal to $-i\overline{d}$ have an additional junction $[d^{-1}-i\overline{d}]$.

Next, notice that the code of the first fraction is d and the code of the last fraction is $-i\overline{d^{-1}}$.

Let us calculate the derived junctions of all the junctions obtained at the first step:

$$[d+i\overline{d}] \mapsto [d^{-1}-d], \qquad [d+d^{-1}] \mapsto [d^{-1}-d], \qquad [d+d] \mapsto [d^{-1}-i\overline{d}],$$
$$[d^{-1}-d] \mapsto [d^{-1}-d], \qquad [d^{-1}-i\overline{d}] \mapsto [d^{-1}-i\overline{d}].$$

Recall that the curves with the eighth-fraction code equal to $-i\overline{d}$ contain the junction $[d^{-1}-i\overline{d}]$, just as all the other junctions, at the first step; hence, the depth of such curves is 1. Since the derived junction of $[d^{-1}-i\overline{d}]$ is a junction of the same form, this junction does not give rise to junctions of different types, the depth of the two remaining curves is 2. $\square$

Theorem 4. *The SLR of the curves in the class M is $5\frac{2}{3}$.*

The proof of this theorem is analogous to the proof of the theorem on the SLR of the minimal N-shaped curve in [4]. It also extensively employs computer simulation. Recall [4] that for a pair of points $p(a)=(A_1,A_2)$ and $p(b)=(B_1,B_2)$ of a Peano curve $p(t)$, the *horizontal* and *vertical* square-to-linear ratios are defined as

$$\frac{(A_1-B_1)^2}{|b-a|} \qquad \text{and} \qquad \frac{(A_2-B_2)^2}{|b-a|},$$

respectively.

The following lemma summarizes the necessary results of computer simulation.

Lemma 18. *The maximum SLR of pairs of corners of the sixth subdivision of any curve in the class M is $5\frac{2}{3}$, and the maximum horizontal SLR of pairs of corners of the sixth subdivision of these curves is $4\frac{1}{2}$.*

Proof. The program written by me computed[1] the SLR for all pairs of corners of the sixth subdivision for all curves in the class M.

The maximum values $5\frac{2}{3}$ and $4\frac{1}{2}$ are attained at the vertices of a junction of type $[d+d]$, which, as is clear from the proof of Lemma 17, is incident to all curves in the set M. For example, if the code of the sixth fraction is d, then the maximum value $5\frac{2}{3}$ of the SLR is attained on the pair of points with coordinates $(0,1)$ (moment $\frac{1}{2}$) and $\left(\frac{5}{9},\frac{2}{3}\right)$ (moment $\frac{93}{162}$). The squared distance between these points is $\frac{34}{81}$, and the time interval is $\frac{2}{27}$. The maximum of the horizontal SLR Q_x is attained, in particular, on the pair with coordinates $\left(\frac{2}{9},\frac{2}{3}\right)$ (moment $\frac{89}{162}$) and $\left(\frac{5}{9},\frac{2}{3}\right)$ (moment $\frac{93}{162}$). The squared distance between these points is $\frac{1}{9}$, while the time interval is $\frac{4}{162}$. The horizontal SLR for these points is $Q_x = 4\frac{1}{2}$. $\square$

Lemma 19. *The maximum horizontal and vertical SLRs of the curves in the set M are equal to each other.*

Proof. The proof is similar to the proof of Lemma 23 from [4]. $\square$

[1]The calculations used integer type data and so are exact.

Proof of Theorem 4. Since the depth of any curve in the class M is not greater than 2 (Lemma 17) and these curves have no singular points (since a diagonal transition is excluded), Theorem 6 from [4] implies that the maximum horizontal SLR is attained on some pair of corners of fractions of the sixth subdivision, and is equal to $Q_x = 4\frac{1}{2}$ by Lemma 18.

Next, the upper bound $Q \leq \frac{17}{3}\left(1 + \frac{2}{9^3}\right) < 6$ for the total SLR can be obtained by Lemma 24 from [4] and Lemma 18 of the present paper. Since $Q \geq 5\frac{2}{3}$, we obtain the following inequalities:

$$\log_9 \frac{Q^2}{4(Q - Q_x)} < \log_9 \frac{6^2}{4\left(\frac{17}{3} - \frac{9}{2}\right)} = \log_9 \frac{54}{7} < 1,$$

which, combined with Theorem 7 from [4] (which applies due to Lemma 19), imply that the maximum of the total SLR is attained on the pairs of corners of the fifth subdivision of the minimal Peano curve. □

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