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Lower estimate of the square-to-linear ratio for regular Peano curves

Abstract: We prove that the square-to-linear ratio of a regular Peano curve that maps the unit interval onto the unit square is at least five.

Keywords: fractals, Peano curve, numerical fractal invariants, square-to-linear ratio.

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1 Introduction

According to Shchepin [1], a *regular Peano curve* is a mapping of a closed interval onto the square, for which the domain may be subdivided into several equal closed intervals (the fractal periods) so that the restriction of the curve to any of its fractal periods is similar to the entire curve. The parts of the curve corresponding to the fractal periods are called *fractions*. The minimal number of isometric fractions into which the curve is divided is called the *fractal genus* of the curve.

In order to apply Peano curves one needs to know some properties thereof. For example, in applications requiring a traverse (screening) of a multi-dimensional lattice it is usually necessary to know how far is a curve from a given point in some time interval. Several approaches for different curves are available in various settings (see [6],[7],[8]). One of these methods involves the so-called *square-to-linear ratio* defined by

$$\frac{|p(t) - p(\tau)|^2}{|t - \tau|}$$

for a pair $p(t)$, $p(\tau)$ of Peano points on a curve $p : [0, 1] \to [0, 1]\times[0, 1]$. The supremum over all square-to-linear ratios of all distinct points on a curve is called the *square-to-linear ratio* of the curve.

For regular fractal Peano curves, the difference $|t - \tau|$ is proportional to the area of the image of the interval $[t, \tau]$. For curves mapping the unit interval onto the unit square (the so-called square curves) this proportionality coefficient equals to 1; that is, such curves are *parametrized by the area*. Moreover, this property is also preserved for their fractions. Hence, in what follows, all the curves will be assumed to be parametrized by the area in estimating the square-to-linear ratio.

In [2] it was shown that the square-to-linear ratio of the Peano–Hilbert curve is six. A set of Peano curves with the smallest square-to-linear ratio in the class of regular Peano curves of fractal genus 9 is put forward in [3] and [5], where this ratio was estimated to be $5\frac{2}{3}$ by numerical analysis.

In [6], the maximum of the square-to-linear ratio is shown to be at least 4 for a wide class of curves, including the regular fractal Peano curves.

Theorem 3 of [1] erroneously states that the initial and final points of any regular fractal square Peano curve are contained in the set of vertices of the image square. The proof, as given in [1], is correct only for cases when the second construction step of the curve does not involve the operation of time reversal.

A counterexample to Theorem 3 is depicted in Fig. 1 showing a traverse at the second construction step of a curve of fractal genus 16, whose initial point is at a vertex of the square, while the final point is in the middle of a side.

Theorems 4 and 5 of [1] state that the square-to-linear ratio of regular fractal Peano curves with initial and final points at vertices of the square may not be smaller than 5.

In the present paper we prove a theorem which completes the results of [1] and justifies the lower bound, which equals to 5, for all regular square Peano curves.

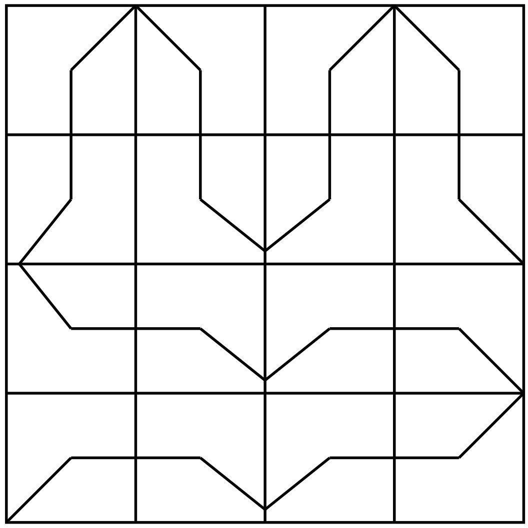

Figure 1. Counterexample to Theorem 3 of [1].

The paper is organized as follows: in § 2 we prove a theorem on the position of the initial and finial points for any regular square Peano curve; § 3 is dedicated to the lower estimate of the square-to-linear ratio for curves with initial points in a vertex of the square and final point in the middle of the opposite side. Finally, in § 2 we formulate and justify the main theorem.

2 Position of end-points of regular Peano curves

Lemma 1. *For any regular Peano curve at least one of the initial and final points lies in a vertex of the square.*

Proof. Similar ideas were employed by Shchepin [1] in the proof of Theorem 3.

Assume that both the initial and final points of the curve do not lie in a vertex of the square. Hence, they may lie either on one side (1), on the neighbouring sides of the square (2) or on the opposite sides of the square (3).

The case (1) is impossible, because in this case the first fraction of the second step has the entrance and exit are at the square boundary and so cannot intersect with other fractions.

The case (3) is also impossible, because in traversing the curve at the second construction step one cannot make a turn, and hence there will be no way to visit all the fractions.

Let us now examine the case (2). Assume that the initial and final points of the curve are at neighbouring sides; then any fractions such that their order numbers along the traverse differ by 2 have nonempty intersection. Consider the corner fraction at the third construction step such that its distance from the initial and final fractions of the curve is larger than 2. In this case, it should have nonempty intersections with the fractions whose order numbers along the traverse differ from by at most 2. In total, there are 4 such fractions, whereas the corner fraction in question has only 3 neighbours. This gives a contradiction. □

Theorem 1. *For any regular Peano curve the initial and final points are either at vertices of the square or one is at a vertex and the other one is in the middle of the opposite side.*

Proof. By Lemma 1, at least one of the initial and final points lies in a vertex of the square. Assume that the initial point lies in the bottom left vertex of the square; otherwise it suffices to consider the curve after reversing the time and making an appropriate rotation.

Assume that the final point lies neither in a vertex, nor in the middle of a side of the square.

The case when the final point lies on the same side as the initial point is impossible, because otherwise in the first fraction the entrance and exit would be on the square boundary, which has no intersection with other fractions.

Assume now that the final point lies on the opposite side from the initial point, for example, on the right vertical side of the square and not at the middle of this side (that it, its distance from the lower boundary of the square is x, $x \notin \{0, \frac{1}{2}, 1\}$). In this case, at the second construction step, any fraction of the same orientation as the entire curve may be followed only by the fraction mirrored to over the given one with respect to their common boundary. Next, the first fraction, as well as the entire curve, has initial point at a vertex, and hence, the final point of the second fraction also lies in a vertex of the partition. Consequently, all odd numbered fractions in the traverse have initial points at vertices of partitions, and for even numbered fractions the final points lie in vertices.

Let the origin of coordinate system be at the lower left corner of the square and the axes be directed along the sides of the square, the length of the side of the fraction at the second construction step of the curve we take as unity.

Now the coordinates of the final point of the second fraction are of the form $(2, 0)$ or $(0, 2)$, and the coordinates of any initial or finite point of the fraction that is in a vertex of the partition are even numbers.

Assume that the curve has an even fractal genus; then the upper left fraction has its initial or final point in the upper left vertex of the square. The curve may not terminate at this fraction, but, on the other hand, this fraction may not have two neighbours. This gives a contradiction.

Assume that curve has an odd fractal genus. Then the upper right fraction starts at its lower left corner and ends on the side of the large square that has no intersections with other fractions. Hence, such a fraction may be final only. But in this case the similarity mapping of the entire curve onto this fraction has a unique fixed point in a corner of the square. Hence, the final point of the curve lies in the upper right corner of the square. This contradicts the initial assumption.

□

3 Lower estimate for special curves

Quintic curves. Following Shchepin [1], a regular square Peano curve is called *quintic* if it is similar to the unit Peano curve and if its square-to-linear ratio is smaller than 5.

Let K denote the class of regular square Peano curves with the initial point in a vertex of the square and the final point in the middle of the opposite side. All proofs will be carried out for the class K, taking into account that curves with initial point at the middle of a side and final point at a vertex of the square are identical to curves from K up to time reversal.

Theorem 2. *The class K of curves does not contain quintic curves.*

For the proof of the theorem we need a few lemmas.

Lemma 2. *For any curve from K any two neighbouring fractions have opposite time-orientations.*

Proof. The concept of a regular Peano curve implies a regular partition of the square, and hence for such a curve no junction between a vertex of the square and the middle of the side of a neighbouring square is possible. Consequently, two neighbouring fractions of the curve from the class K may not have the same time-orientation. □

Consider the junction of two fractions, of which the first one is initial (that is, its time-orientation agrees with the orientation of the entire curve [1]), and the second one has reverse time-orientation (by Lemma 2). By suitable rotations and reflections of the curve we may put the initial point of the first fraction to its lower left corner and the final point to the middle of its right side. Hence, the second fraction emerges from the middle of the left side and ends at a vertex on the right side. If its final point lies in the bottom right vertex (respectively, upper right vertex), then such a junction will be called mirror (respectively, inverted).

Lemma 3. *Any curve from the class K has an inverted junction.*

Proof. Assume that the initial point lies in the bottom left vertex of the square and the final point lies in the middle of the right side; otherwise it suffices to consider the curve after a suitable rotation.

Let the origin of coordinate system be at the lower left corner of the square, the axes be directed along the sides of the square, the length of the side of the fraction at the second construction step of the curve will be taken as unity.

Assume that the curve has only mirror junctions.

The coordinates of the final point of the second fraction are either (2, 0) or (0, 2). We also note that the coordinates of any final or initial point of a fraction lying in a vertex of the partition are even numbers.

Assume that curve has an even fractal genus. Then the initial or final point of the upper left fraction is in the upper left vertex of the square. This point may not be the end-point of the entire curve. On the other hand, it may not have two neighbours in the traverse. Consequently, a traverse with such conditions is impossible.

Assume that curve has an odd fractal genus. Then the upper right fraction starts at its lower left corner and ends on the side of the large square, which has no intersections with other fractions. However, such a fraction may not be final by the defining condition of the class K. Hence, such a traverse is also impossible.

This being so, it is shown, regardless of the parity of the fractal genus of a curve, that it cannot be constructed by means of specular junctions only. □

Lemma 4. *Quintic curves from the class K do not contain diagonal junctions.*

Proof. Assume that a quintic curve from the class K has a diagonal junction.

Consider possible variants of relative positions of fractions at the next step with a diagonal junction (Fig. 2). The first and second fractions (counting from the corner) always have a common side, and since the first fraction emerges from a vertex, it follows that a transition into the second fraction takes place through the middle of the side.

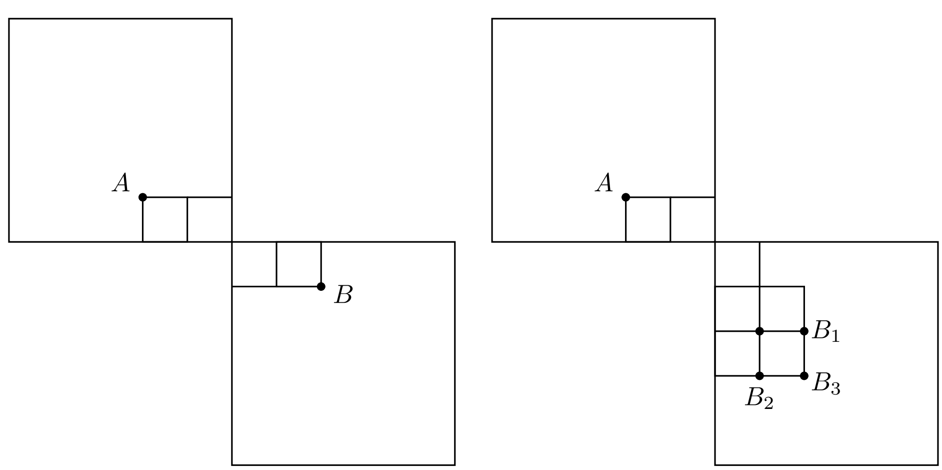

Figure 2. The diagonal junction.

In one case (Fig. 2, on the left), it suffices to consider two pairs of fractions at the next step (counting from the vertex of the junction) and find a lower estimate of the square-to-linear ratio between the marked points:

$$F(A, B) \geq \frac{2^2 + 4^2}{4} = 5\,.$$

In the second case (Fig. 2, on the right), we note that the transition between the second and third fractions (counting from the corner) takes place in a vertex. Hence, there exist three fractions which may be third in the traverse from the corner. In all three cases we consider the farthest vertex of the third fraction (counting from the vertex of the diagonal junction). In Fig. 2 (on the right) these points are denoted B_1, B_2, B_3. At these points the square-to-linear ratio is estimated as follows:

$$F(A, B_i) \geq \frac{3^2 + 4^2}{5} = 5\,.$$

In both cases we arrive at a contradiction with the fact that the curve is quintic. □

Lemma 5. *The class K contains no quintic curves of even fractal genus.*

Proof. Consider a curve of even fractal genus from the class K. By Lemma 2, the orientations of fractions at the second construction step alternate. Since the first fraction is initial, it follows that the last fraction has opposite time-orientation; that is, the curve emerges from the last fraction in its vertex. Consequently, by Lemma 3, the curve has an inverted junction, and hence, this junction at the next construction step of the curve will have a diagonal junction. Such a curve may not be quintic by Lemma 4. □

Lemma 6. *The class K contains no quintic curves of odd fractal genus.*

Proof. Assume the contrary: the class K contains at least one quintic curve of odd fractal genus.

By Lemma 2, the orientations of fractions at the second construction step alternate and the first fraction is initial, and hence the last fraction is also initial; that is, the curve emerges from the last fraction in the middle of its side.

Recall that by Lemma 3 the curve has an inverted junction. Consider this junction in more detail. In Fig. 3 we show a junction in which the left fraction starts in the lower left corner, the right fraction ends in the upper right corner, and the transition occurs in the middle of its common side. The figure also shows some fractions of the next construction step of the curve. For convenience of exposition, small fractions are numbered.

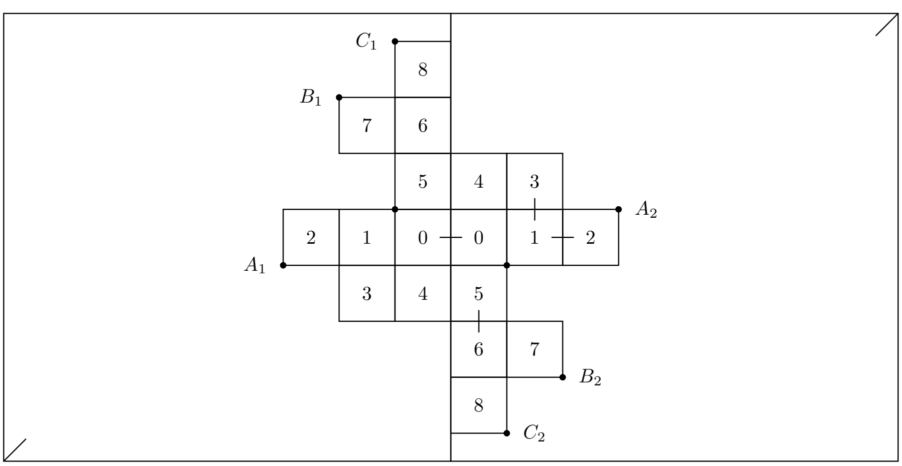

Figure 3. A inverted junction for a curve of odd fractal genus.

The fractions with number 0 are, respectively, the last small fraction in the left fraction and the first small fraction in the right fraction; the transition between them occurs in the middle of their common side. The curve has an odd fractal genus, and hence the first and last fraction of the curve are initial. Therefore, in a fraction with number 0 transitions to neighbours occur only in vertices of the partition.

We shall consider the traverse of the curve in the right fraction, because the traverse in the left fraction is symmetric.

Assume that the exit from the fraction with number 0 takes place in the lower right corner, for otherwise we may invert the junction with respect to the horizontal line and continue the proof.

Recall that: 1) the curve is quintic by the assumption, and hence, it does not have diagonal junctions by Lemma 4; 2) the junction between the second and third (along the traverse) fractions occurs in the middle of the side by Lemma 2. Hence, the following traverses through the subfractions of the right fraction at the junction are possible: $(0, 1, 2)$, $(0, 1, 3)$, $(0, 5, 6, 7)$, $(0, 5, 6, 8)$.

In the case $(0, 1, 2)$ we have

$$F(A_1, A_2) \geq \frac{6^2 + 1}{6} > 5\,.$$

This contradicts that the curve is quintic.

In the case $(0, 1, 3)$ we note that fraction 4 may not be traversed. Hence, such a traverse is impossible.

In the cases $(0, 5, 6, 7)$ and $(0, 5, 6, 8)$ the square-to-linear ratio is estimated for the respective pairs of points as follows:

$$F(B_1, B_2) \geq \frac{4^2 + 5^2}{8} > 5; \qquad F(C_1, C_2) \geq \frac{7^2 + 2^2}{8} > 5\,.$$

Thus, all the traverses satisfying the assumption are examined, so it is proved that in any case the square-to-linear ratio is at least 5 for a curve from the class K of odd fractal genus. □

Proof (*of Theorem 2*). The class K of curves consists of the curves of even and odd fractal geni. However, by Lemmas 5 and 6 they all fail to contain quintic curves. □

4 Lower bound for the square-to-linear ratio of regular Peano curves

Theorem 1 asserts that any regular Peano curve has the initial and final points either at vertices of the square or one point is at a vertex and the other one is in the middle of the opposite side. Theorems 4 and 5 of [1] assert that, for a regular Peano curve with the initial and final points at vertices of the square, the square-to-linear ratio is at least 5.

Theorem 2 completes the results of [1] and justifies the following result.

Theorem 3. *Each regular square Peano curve has square-to-linear ratio at least* 5.

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